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JOURNAL OF INVESTMENT MANAGEMENT, Vol. 2, No. 3, (2004), pp. 1–15

ACTIVE RISK AND INFORMATION RATIO

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One of the underlying assumptions of the Fundamental Law of Active Management is that the active risk of an active investment strategy equates estimated tracking error by a risk model. We show there is an additional source of active risk that is unique to each strategy. This strategy risk is caused by variability of the strategy's information coefficient over time. This implies that true active risk is often different from, and in many cases, significantly higher than the estimated tracking error given by a risk model. We show that a more consistent *estimation of information ratio is the ratio of average information coefficient to the standard deviation of information coefficient. We further demonstrate how the interaction between information coefficient and investment opportunity, in terms of cross sectional dispersion of actual returns, influences the IR. We then provide supporting empirical evidence and offer possible explanations to illustrate the practicality of our findings when applied to active portfolio management.*

1 Introduction

Information ratio (IR), the ratio of average excess return to active risk, is an important performance measure for active investment management. One result regarding *ex ante* IR is Grinold's (1989) Fundamental Law of Active Management, which states that the expected IR is the expected information coefficient (IC) times the square root of breadth. IC refers to the cross-sectional correlation coefficient between forecasts of excess returns and actual returns. For equity portfolios—the focus of the present paper, the breadth is the number of stocks within a select universe. In mathematical terms, the relationship is

$$\mathrm{IR} = \overline{\mathrm{IC}}\sqrt{N} \tag{1}$$

Throughout the paper, the bar denotes the expected value.

Equation (1), while providing insight to active management, is based on several simplified assumptions. Various studies re-examine this result when different assumptions are used. For instance, one of

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the assumptions is that active portfolio is a pure long–short portfolio free of long-only constraint. Grinold and Kuhn (2000) examine how IR deviates from Eq. (1) under the long-only and other portfolio constraints using simulation techniques. Recently, Clarke *et al.* (2002) developed a framework for measuring such deviations by including a "transfer coefficient" on the right-hand side of Eq. (1). In addition to the long-only constraint, they also study the impact of constraints in terms of turnover as well as factors such as size and style. Both studies conclude that portfolio constraints generally tend to lower *ex ante* IR, as given in Eq. (1).

Equation (1) hinges on another simplified assumption regarding active risk of investment strategies. Namely, it assumes that the active risk of an investment strategy is identical to the tracking error estimate by a risk model. Our research shows that *ex post* active risk often significantly exceeds the target tracking error by risk models, even after appropriately controlling risk exposures specified by a risk model. In this paper, we will unveil an additional source of active risk that accounts for this discrepancy. This new source of risk stems from the variability of IC, i.e. the correlation between forecasts and actual returns. Hence, it is unique to each investment strategy and we shall refer to it as strategy risk. Mathematically, it is the standard deviation of IC, i.e. std(IC).

The previous research mentioned above, while acknowledging the average IC of different strategies, assumes that all strategies have the same active risk if they have the same target tracking error. This simplified assumption is not adequate in characterizing different investment strategies. As we will show below, the true active risk is a combination of the risk-model risk and the strategy risk. Although there are other alternative measurements of active risk, we consider standard deviation of excess return or tracking error in the present paper. We use active risk and tracking error interchangeably.

It is no surprise that the variability of IC plays a role in determining the active risk. Just imagine two investment strategies, both taking the same risk-model tracking error σ_{model} over time. The first strategy is blessed with perfect foresight and it generates constant excess return every single period. In other words, it has a constant positive IC for all periods such that std(IC) is zero. Such a risk-free strategy, admittedly hard to find, has constant excess return, and thus, no active risk whatsoever. However, the risk model is not aware of the prowess of the strategy and dutifully predicts the same tracking error all the time. In this case, the risk model undoubtedly overestimates the active risk. In contrast, the second strategy is extremely volatile with large swings in its excess return, i.e. its IC varies between -1 and $+1$ with a large std(IC). As a result, its active risk might be much larger than the risk model estimate. Thus, the two strategies with identical risk-model tracking error have very different active risk in actuality.

In practice, the difference between active investment strategies is not that extreme. However, our experience shows that risk-model tracking error given by most commercially available risk models routinely and seriously underestimates the *ex post* active risk.[1] This underestimation could have serious practical consequences. For example, an enhanced index product with low risk-model tracking error but high standard deviation of IC could be far more risky, because the true active risk is larger.

Our results will enable portfolio managers to obtain more accurate estimates of active risk of their active strategies, and as a result, better estimates of IR. Furthermore, they can be used jointly with the results of Grinold and Kuhn (2000) and Clarke *et al.* (2002) by portfolio managers to provide realistic IR estimates.

2 Notations and main results

To facilitate our analysis, we introduce the following notations and terminologies.

- *Risk-model tracking error*, *denoted as* σ_{model}: It is the tracking error or the standard deviation of excess returns estimated by a generic risk model such as BARRA, and it is also referred to as *risk-model risk* or *target tracking error*.
- *Strategic risk*, *denoted as* std(IC): It is the standard deviation of IC of an investment strategy over time. It is unique to each active investment strategy, conveying strategy-specific risk profile.
- *Active risk*, *denoted as* σ: It is the active risk or tracking error of an investment strategy measured by the standard deviation of excess returns over time.

Our main result regarding the active risk is the following: the active risk is a product of the strategy risk, the square root of breadth, and the risk-model tracking error:

$$\sigma = \text{std(IC)}\sqrt{N}\sigma_{\text{model}} \tag{2}$$

This result has several clear implications. First, the active risk is *not* the same for different investment strategies due to varying levels of strategy risks. Second, only rarely does the active risk equal the risk-model tracking error. It happens only when strategy risk, std(IC), is exactly equal to the reciprocal of the square root of N. This is true in an ideal situation, in which the standard deviation of IC is proportional to the sampling error of a correlation coefficient, which is the reciprocal of the square root of N. In reality, however, as our empirical results will show, the standard deviation of IC bears little relationship to this theoretical sampling error, and is significantly different for different strategies.

We note that our paper is not a critique of any risk model because our focus is not the same as studying the measurement error of risk models over a single rebalancing period. In those studies (e.g. Hartmann *et al.*, 2002), one analyzes the performance of risk models over a single, relatively short period, during which the examined portfolios are bought and held. The approach is to compare predicted tracking errors of a risk model to the realized tracking errors using either daily or weekly excess returns for many simulated portfolios. Hartman *et al.* (2002) attribute the difference between the estimated risk and the *ex post* tracking error to several items: estimation error in covariances in a risk model, time varying nature of covariances, serial auto-correlations of excess returns, and the drift of portfolio weights over a given period. Depending on how these factors play out in a given period, a risk model can overestimate as well as underestimate with seemingly equal probability *ex post* tracking errors of simulated portfolios. There is no clear evidence of bias one way or the other.

In contrast, we study the active risk of an investment strategy over multiple rebalancing periods, during which the active portfolio is traded periodically based on the forecasts of that investment strategy. While it is useful to consider the single-period active risk of a buy-and-hold portfolio, it is arguably more practical to analyze the active risk over multiple rebalancing periods. Our analysis reveals a clear underestimation bias of risk-model risk even if the risk model is adequate. This is because using a risk model alone is not enough to accurately estimate the true active risk. Only through consideration of strategy risk can an unbiased estimate of active risk be obtained.

Because of more realistic estimate of active risk, our estimate of IR is different from that of Eq. (1). We shall show that IR of an investment strategy is

$$\text{IR} = \frac{\overline{\text{IC}}}{\text{std(IC)}} \tag{3}$$

Equation (3) is very intuitive. Since IR measures the ratio of average excess return to the standard deviation of excess return, if IC were the sole determinant of excess return, then IR would be the ratio of average IC to the standard deviation of IC. In most of the cases we have studied, IR is lower than that of Eq. (1) because the true active risk tends to be higher than the risk-model tracking error.

3 Cross-sectional IC and single-period excess return

To derive the IR of an active investment strategy over multiple periods, we start by calculating a single-period excess return, which is the summed product of active weights and subsequent realized actual returns. We use active mean–variance optimization to derive the active weights under the following framework. First, we model security risk by a generic multi-factor fundamental risk model, such as the BARRA risk model. Second, the optimal active weights are selected by mean–variance optimization while neutralizing portfolio exposures to all risk factors, in addition to being dollar neutral. We have done so for two reasons. First, the alpha factors we shall study in the empirical section below are employed by quantitative managers mostly to exploit stock specific returns. The second reason is more technical. Imposing binding constraints on all risk factors allows us to derive an analytical solution for the optimal portfolio weights without knowing the historical covariance matrices of risk factor returns. While it is certainly possible to extend our analysis to strategies that also take factor bets, the research is out of the scope of this article. While we reasonably expect that different factor-related strategies would have their own component of strategy risk, practitioners should use caution when applying our results directly to those strategies.

Under these conditions, Appendix A gives the exact solution for the active weights $w_{i,t}$ for security i and time t. The excess return for the period is the summed product of the active weights $w_{i,t}$ and the subsequent actual return $r_{i,t}$. To reflect dollar and factor neutral constraints, we recast the summed product expression by adjusting both the forecasts and the actual returns to obtain

$$\alpha_t = \lambda_t^{-1} \sum_{i=1}^{N} R_{i,t} F_{i,t} \tag{4}$$

where λ is a risk-aversion parameter used in the optimization, R is the risk-adjusted actual return, and F is the risk-adjusted forecast. They are the "raw" return or forecast adjusted for dollar and factor neutrality, and then normalized by security specific risk (Appendix A).

So far, our derivation of Eq. (4), in Appendix A, has been standard. Similar analyses can be found in Grinold (1989) and Clarke *et al.* (2002). From this point on, our analysis uses a different approach. In previous work (Grinold, 1994; Clarke *et al.*, 2002), one makes an assumption about the expected returns of individual securities, such as "Alpha is Volatility Times IC Times Score" (Grinold, 1994). The validity of such a normative approach, which has its origin in risk modeling, is questionable in reality. We shall adapt a descriptive approach with no assumptions regarding individual securities.[2] We write Eq. (4) as the covariance between the risk-adjusted returns and forecasts, which in turn can be rewritten as a product of IC and their dispersions.[3] We have

$$\begin{aligned} \alpha_t &= \lambda_t^{-1}(N-1)[\mathrm{cov}(\mathbf{R}_t, \mathbf{F}_t) + \mathrm{avg}(\mathbf{R}_t)\mathrm{avg}(\mathbf{F}_t)] \\ &= \lambda_t^{-1}(N-1)\mathrm{IC}_t \mathrm{dis}(\mathbf{R}_t)\mathrm{dis}(\mathbf{F}_t) \end{aligned} \tag{5}$$

We use $\mathbf{R}_t$ and $\mathbf{F}_t$ to denote the cross-sectional collections of the risk-adjusted returns and forecasts, and $\mathrm{IC}_t = \mathrm{corr}(\mathbf{R}_t, \mathbf{F}_t)$. The average term in Eq. (5) vanishes because we have made $\mathrm{avg}(\mathbf{R}_t) = 0$ (see

Appendix A). Equation (5) states that the single-period excess return is proportional to the IC of that period and the dispersions of the risk-adjusted returns and forecasts for that period. The intuition is clear: the excess return is a function of IC, which measures the forecast's cross-sectional ranking ability, the dispersion of the forecasts, which reflects the perceived cross-sectional opportunity, and the dispersion of the actual returns, which represents the actual cross-sectional opportunity.

The risk-model risk, on the other hand, depends only on the dispersion of the forecasts through the optimal active weights. They are related by (see Appendix A)

$$\sigma_{\text{model}} \approx \lambda_t^{-1}\sqrt{N-1}\,\text{dis}(\mathbf{F}_t) \tag{6}$$

In other words, the risk-model risk is the dispersion of the risk-adjusted forecasts (which varies from period to period) times the square root of $N-1$ divided by the risk-aversion parameter. Equations (5) and (6) show that, while the excess return depends on IC and both dispersions, the risk-model risk is only a function of the forecast dispersion. In other words, the risk-model risk is independent of IC since the risk model has no knowledge of the information content of the forecasts.

We shall maintain a constant level of risk-model tracking error[4] by varying the risk aversion parameter accordingly. Combining Eqs. (5) and (6) produces the relationship

$$\alpha_t \approx \text{IC}_t\sqrt{N}\sigma_{\text{model}}\text{dis}(\mathbf{R}_t) \tag{7}$$

We have replaced $N-1$ with N, which is justified when N is large enough. The excess return of an active strategy in a single period is IC times the square root of breadth times the risk-model tracking error times the dispersion of the risk-adjusted returns. Among the four terms in Eq. (7), the dispersion of the risk-adjusted returns is new and thus deserves some discussion. In theory, if the risk model truly describes the return of every single security, then each risk-adjusted return $R_{i,t}$ is close to a standard normal random variable. The base case estimation for the dispersion of a large number of such random variables is unity.[5] Later, we shall see that this is approximately true for certain risk models. This dispersion represents the degree of opportunity in the market. For a given level of IC and risk-model risk, a greater opportunity leads to a higher excess return.

4 Information ratio

We derive IR of an investment strategy over multiple periods. Equation (7) is close to a mathematical identity. While it is always true *ex post*, we now use it in *ex ante* by considering its expectation and standard deviation, i.e. the expected excess return and the expected active risk. Among the four terms affecting the excess return, we assume that the number of stocks does not change over time and the risk-model tracking error remains constant. For the two remaining terms that do change over time, IC is associated with greater variability than the dispersion of the risk-adjusted returns. Therefore, as a first approximation we treat the latter also as a constant.

4.1 The simple case

Assuming $\text{dis}(\mathbf{R}_t)$ is constant and equal to its mean, the expected excess return is

$$\bar{\alpha}_t = \overline{\text{IC}_t}\sqrt{N}\sigma_{\text{model}}\overline{\text{dis}(\mathbf{R}_t)} \tag{8}$$

The expected excess return is, therefore, the average IC (skill) times the square root of N (breadth) times the risk-model tracking error (risk budget) times the dispersion of actual returns (opportunity).

The expected active risk is

$$\sigma = \text{std}(\text{IC})\sqrt{N}\sigma_{\text{model}}\overline{\text{dis}(\mathbf{R}_t)} \tag{9}$$

The standard deviation of IC measures the consistency of forecast quality over time. Therefore, the active risk is the standard deviation of IC (strategy risk) times the square root of N (breadth) times the risk-model tracking error (risk budget) times the dispersion of actual returns (opportunity).

The ratio of Eq. (8) to (9) produces Eq. (3); i.e. IR is the ratio of the average IC to the standard deviation of IC, or IR of IC. We also note that when the mean dispersion is unity, Eq. (9) reduces to Eq. (2).

4.2 A better estimation of IR

In reality, the variability in the dispersion of the risk-adjusted return $\text{dis}(\mathbf{R}_t)$ is small but, nonetheless, non-zero. What happens to IR if we include this variability? The following insight from Eq. (7) helps us to understand how the interaction between the IC and the dispersion affects the excess return. To produce a high positive excess return for a single period, we need high and positive IC as well as high dispersion. Conversely, when IC is negative, we would like a low dispersion so that the negative excess return would be small in magnitude. This argument implies that, over the long run, the performance will benefit from a positive correlation between IC and the dispersion. On the other hand, a negative correlation will hurt the average excess return.

Appendix B shows that the expected excess return including this correlation effect is

$$\bar{\alpha}_t = \sqrt{N}\sigma_{\text{model}}\{\overline{\text{IC}}_t\overline{\text{dis}(\mathbf{R}_t)} + \rho[\text{IC}_t, \text{dis}(\mathbf{R}_t)]\text{std}(\text{IC}_t)\text{std}[\text{dis}(\mathbf{R}_t)]\} \tag{10}$$

The additional term consists of the correlation between IC and the dispersion, and the standard deviations of IC and the dispersion. According to Appendix B, the active risk is little affected by the correlation because the coefficient of variation of $\text{dis}(\mathbf{R}_t)$ is much smaller than that of IC and one. Combining Eqs. (9) and (10) produces the new IR estimate

$$\text{IR} = \frac{\overline{\text{IC}}_t}{\text{std}(\text{IC}_t)} + \rho[\text{IC}_t, \text{dis}(\mathbf{R}_t)]\frac{\text{std}[\text{dis}(\mathbf{R}_t)]}{\overline{\text{dis}(\mathbf{R}_t)}} \tag{11}$$

The second term captures the correlation effect on IR. It has two factors. The first is the correlation between IC and the dispersion over time and the second term is the coefficient of variation of the dispersion. As we mentioned earlier, the coefficient of variation of the dispersion is usually small. Therefore, the effect of the second term is typically small unless the correlation between IC and the dispersion becomes very high, either positive or negative. For most practical purposes, Eq. (3), i.e. the first term in Eq. (11), approximates IR well enough. Nonetheless, Eq. (11) is an improvement.

5 Empirical examinations

To demonstrate that Eq. (9) is a more *consistent* estimator of *ex ante* active risk, we study empirical results of 60 quantitative equity strategies. To ensure practical relevance, these strategies are based on a set of quantitative factors commonly used by active managers. The set encompasses a wide range of well-known market anomalies, and thus provides a good representation of different categories of quantitative strategies deployed by active managers.

We first briefly describe the data. Then, we apply the analysis to the Russell 3000 indices to demonstrate our theoretical result. To assess the statistical significance of the differences in the strategy risk, we provide a closer examination of two valuation factors—gross profit to enterprise value and forward earnings yield. We introduce a strategy-specific scaling constant κ and use it in conjunction with a risk model to provide a *consistent* forecast of *ex post*

active risk. Lastly, we suggest different ways to forecast strategy risk and ascertain the efficacy of such predictions.

5.1 The data

The quarterly data used in our analysis span 1987 to 2003, with 67 quarters in total. The alpha factors come from a proprietary database and they include seven different categories: price momentum, earnings momentum, earnings surprise, valuation, accruals, financial leverage, and operating efficiency. The values for beta, systematic risk factors, industry risk factors, and stock specific risk come from the BARRA US E3 equity risk model. To ensure factor accuracy and to prevent undue influence from outliers, we first exclude stocks that have factor values exceeding five standard deviations on each side. Next, we bring factor values between three and five standard deviations to the three standard deviation values. The actual number of stocks that are tested against the Russell 3000 index is, therefore, fewer than 3000. In addition, the number of stocks fluctuates from quarter to quarter due to data availability as well as the reconstitution activities of Russell indices. However, the fluctuation is insignificant as to alter the analysis.

In terms of portfolio construction, we form optimal long–short portfolios on a quarterly basis. Subsequently, cross-sectional analyses of alpha and IC and dispersion of the risk-adjusted returns are computed on a quarterly basis. We set the constant risk-model tracking error at 2.5% per quarter. Additionally, to control risk exposures appropriately, we neutralize active exposures to all BARRA risk factors (market beta, 13 systematic risk factors, and 55 industry risk factors) when rebalancing portfolios each quarter. Hence, the risk-model risk is 100% stock specific according to the risk model. The results below are collected on a quarterly basis and are annualized for the purposes of this paper. For example, the annualized target tracking error would be 5%, provided there is no serial auto-correlation in alpha.

5.2 The Russell 3000 universe

Figure 1 shows the histogram of *ex post* active risk of the 60 strategies. Although the risk-model

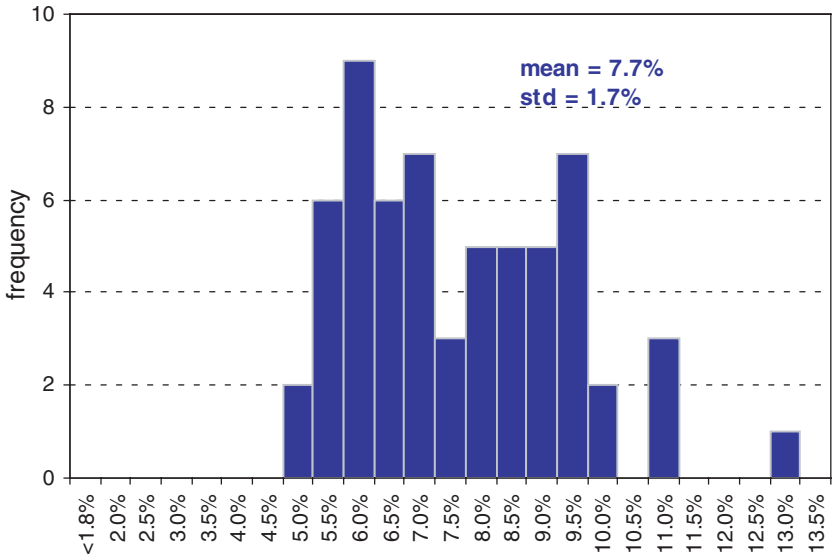

Figure 1 Histogram of the *ex post* active risk of equity strategies.

tracking error is targeted at 5% for all strategies, the *ex post* active risks differ widely with substantial upward bias, indicating the risk model's propensity to underestimate active risk. The average active risk is 7.7% and the standard deviation is 1.7%. The highest active risk turns out to be 13.1% while the lowest is just 5.0%. In other words, almost all strategies experienced *ex post* risk higher than the risk-model tracking error.

To gauge the risk model's estimation bias in relative terms, we rearrange Eq. (9) to derive a scaling constant κ that approximates the ratio of true active risk to the risk-model risk, in terms of the standard deviation of IC for each factor and the average number of stocks over time:

$$\kappa = \text{std(IC)}\sqrt{N} \approx \frac{\sigma}{\sigma_{\text{model}}} \qquad (12)$$

We have neglected the dispersion of returns, $\overline{\text{dis}(\mathbf{R}_t)}$, because it turns out to be very close to unity with a value at 1.01 and a standard deviation of 0.15. By this measure, the BARRA E3 model shows remarkable internal consistency. Figure 2 shows the histogram of the scaling constant κ for all 60 strategies. Note that for a majority of strategies the model underestimates the *ex post* active risk by 50% or more. Figure 2 resembles Figure 1 quite closely except that the x-axis is rescaled by the risk-model tracking error of 5%. A scatter plot of the active risk and κ (Figure 3) confirms the observation. Additionally, Table 1 reports the estimated coefficients of the regression using the scaling constant κ to explain *ex post* active risk. The R-squared of this regression is 98%, indicating that Eq. (9) is a highly accurate approximation of the *ex post* active risk despite the assumption that $\text{dis}(\mathbf{R}_t)$ is constant over time. More importantly, it seems possible that practitioners can use the scaling constant κ to adjust risk-model tracking error to achieve a *consistent* forecast of active risk. We demonstrate this adjustment below.

5.3 *Information content of strategy risk: an example*

The strategy risks of these quantitative strategies vary greatly. Naturally, one wonders about the statistical significance of their differences. These differences are important in terms of forecasting portfolio active risk that incorporates strategy risk.

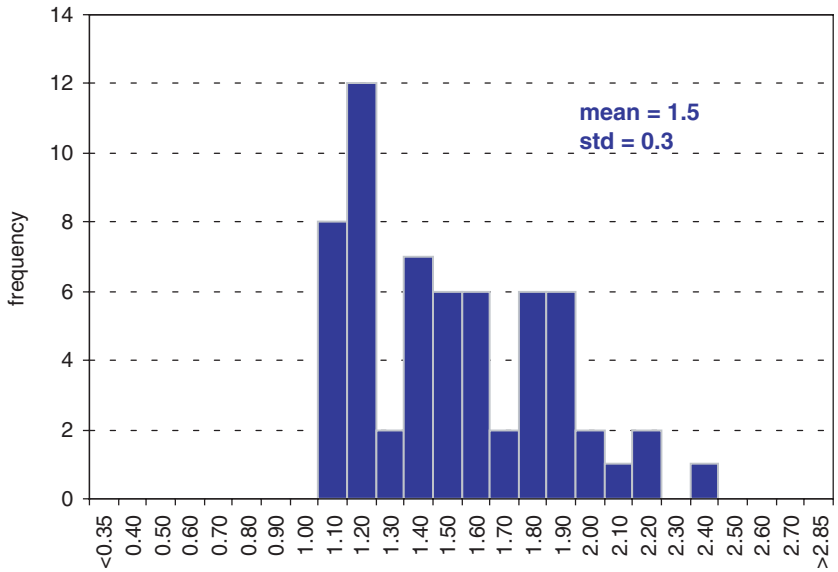

Figure 2 Histogram of the scaling constant κ of equity strategies.

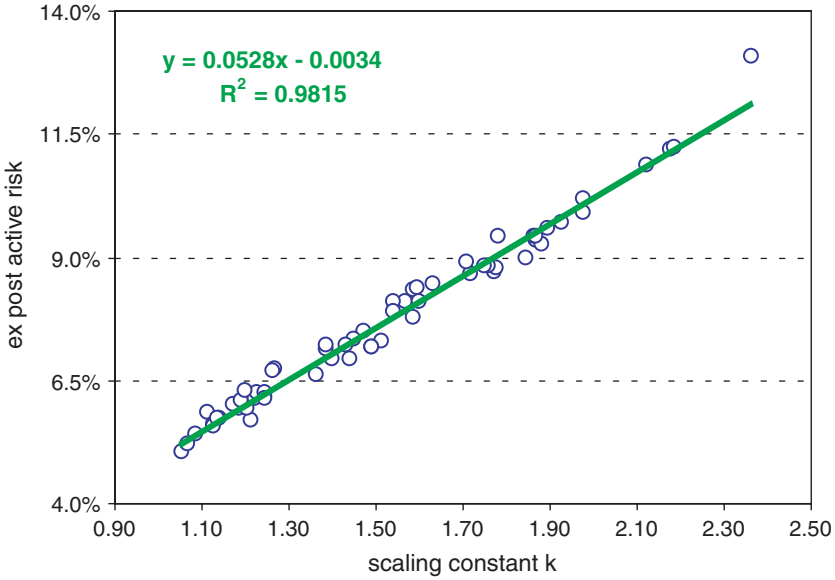

Figure 3 Scatter plot of *ex post* active risk and scaling constant.

Table 1 Summary statistics of coefficient estimates.

	Coefficients	Standard error	*t*-Stat	*P*-value	Lower 95%	Upper 95%
Intercept	−0.0034	0.0015	−2.3260	0.0235	−0.0063	−0.0005
Scaling constant κ	0.0528	0.0009	55.9084	0.0000	0.0509	0.0547

Table 2 Summary statistics of valuation factors.

	Average Alpha	STD of Alpha	IR of Alpha	Average IC	STD of IC	IR of IC	Average dis(*R*)	Average *N*
GP2EV	6.2%	6.9%	0.90	2.4%	2.7%	0.91	1.01	2738
E2P	3.3%	8.7%	0.38	1.4%	3.4%	0.41	1.00	2487

In other words, after appropriately controlling risk exposures specified by the BARRA E3 model in our case, does the standard deviation of ICs provide additional insight regarding the risk profile of an equity strategy? The answer to this question is "yes" in many cases. Here we select two valuation factors—gross profit to enterprise value (GP2EV) and forward earnings yield based on IBES FY1 consensus forecast (E2P)—for a closer examination. We test the statistical significance of the difference between the two strategy risks using the *F*-test.

Table 2 shows the summary statistics of these two factors. For GP2EV, the standard deviation of IC equals 2.7%; it is 3.4% for E2P. The *ex post* tracking errors are 6.9% and 8.7%, respectively. Since both standard deviations are estimated over 67 quarters, the degree of freedom equals 66. The variance ratio of the two factors is $(3.4 \times 3.4)/(2.7 \times 2.7) = 1.58$ and α equals 0.032. Thus, in this example, there is enough evidence to reject the null hypothesis that these two factors, from the same valuation category, have the same strategy risk at a 5% confidence level.

Our results indicate that the strategy risks of factors selected from different categories, more often than not, are statistically different.

5.4 *Consistent estimator of active risk*

Can practitioners use strategy risk in conjunction with a risk model to compute a more *consistent* active risk forecast? As a first attempt to answer this question, we divide the testing period into two halves: in-sample period (1986–1994) and out-of-sample period (1995–2003). In the in-sample period, we estimate κ according to Eq. (12) for each of the 60 equity strategies. Then, in the out-of-sample period, we adjust the risk-model tracking error by $1/\kappa$, using strategy-specific κ to compensate the risk model's bias in estimating active risk. The *adjusted* risk-model tracking error is

$$\sigma^*_{\text{model}} = \frac{\sigma_{\text{model}}}{\kappa} \tag{13}$$

Figure 4 shows the distribution of *ex post* active risks in the out-of-sample period when we set the target tracking error at 5%/κ (the *adjusted* risk-model tracking error), and for comparison, Figure 5 shows active risk of portfolios targeting the same tracking error at 5% (the original risk-model tracking error). We would like to emphasize again that the *adjusted* risk-model tracking error σ^*_{model} is unique to each equity strategy depending on its κ estimates, while the risk-model tracking error σ_{model} is the same for all strategies. From these two histograms, it is obvious that σ^*_{model} is a more *consistent estimator* of active risk. First, the average *ex post* active risk is 4.7% when using σ^*_{model}, and 7.6% when using σ_{model}. Thus, the expected *ex post* active risk is much closer to our target of 5% with no bias when using the *adjusted* risk-model tracking error. Second, the *adjusted* risk-model tracking error also provides a tighter, more bell-shaped distribution of *ex post* active risks. The standard deviation of *ex post* active risk is 0.76% when using σ^*_{model}, and 1.45% when using σ_{model}. It is apparent that in this shorter period, the risk model experienced a similar problem of underestimating the true active risks of many strategies.

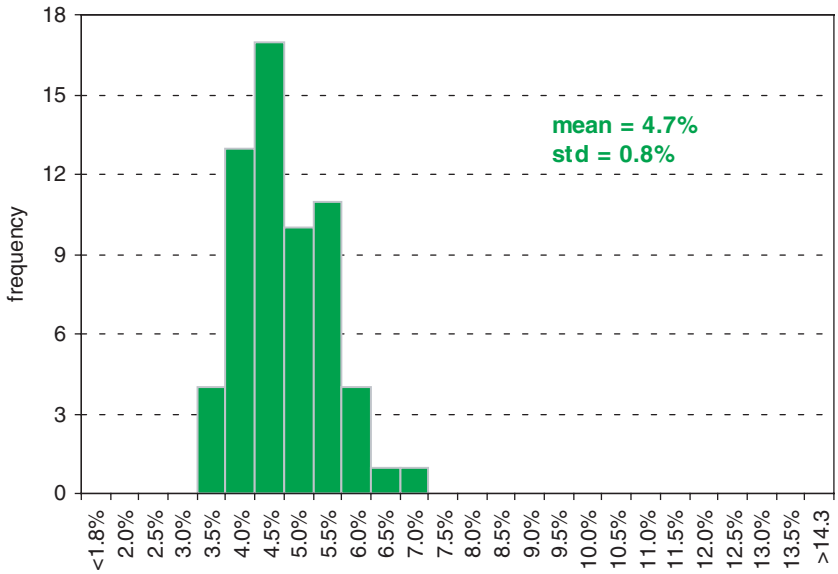

Figure 4 Histogram of the *ex post* active risks using adjusted model TE (1995–2003).

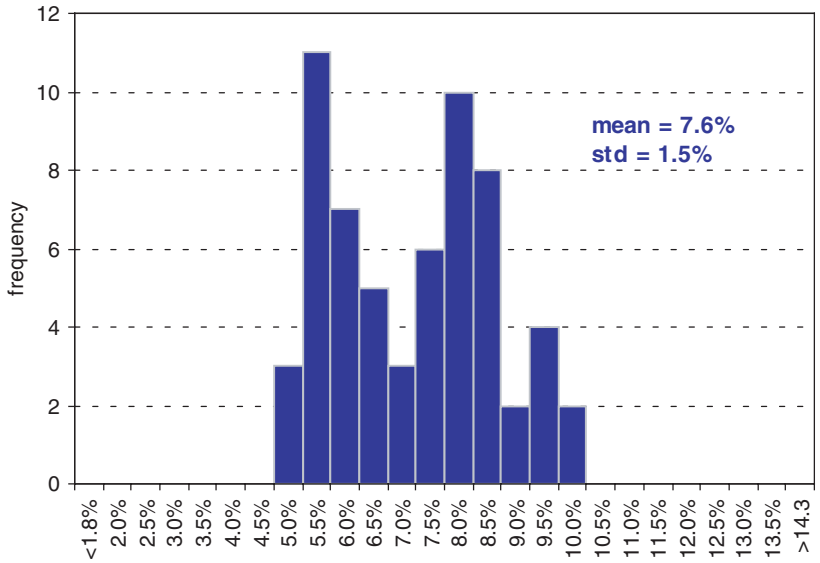

Figure 5 Histogram of the *ex post* active risks using 5% Model TE (1995–2003).

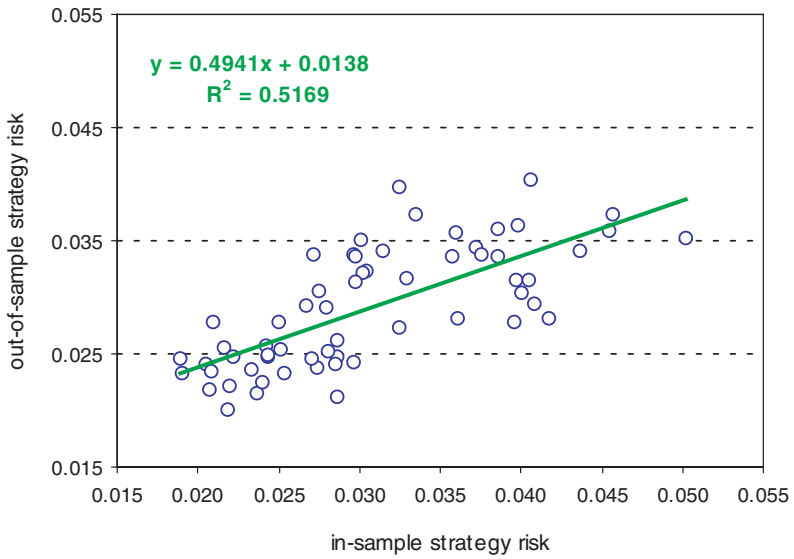

Figure 6 Scatter plot of in-sample strategy risk versus out-of-sample strategy risk.

5.5 Persistence of strategy risk

Naturally, one must be able to forecast the strategy risk, std(IC), with *reasonable* accuracy in order to provide a consistent forecast of active risk using Eq. (9). The application of the scaling constant κ above constitutes a simplistic form of forecasting strategy risk—using the strategy risk estimated in the in-sample period as the forecast of the out-of-sample period. Our simplistic forecasting method assumes that strategy risk persists from the in-sample period to the out-of-sample period. One implication of this methodology is that the relative ranking of strategy risks stays the same in both periods. We employ the in-sample and out-of-sample specification to show this is indeed the case.

Figure 6 shows the scatter plot of strategy risks measured in the in-sample period (x-axis) versus that in the out-of-sample period (y-axis). The R-squared

Table 3 Summary statistics of coefficient estimates.

	Coefficients	Standard error	*t*-Stat	*P*-value	Lower 95%	Upper 95%
Intercept	0.0138	0.0020	7.0245	0.0000	0.0099	0.0178
In-sample strategy risk	0.4941	0.0622	7.9449	0.0000	0.3697	0.6186

of the regression, using in-sample strategy risks to explain the variability of out-of-sample strategy risks, is 52%. Table 3 shows the summary statistics of the coefficient estimates of this regression. The null hypothesis, that in-sample strategy risks have no explanation power of the variability of the out-of-sample strategy risks, is rejected at a 1% confidence level. Hence, it is plausible that, using this simplistic forecast method in conjunction with Eq. (9), active managers can improve their ability to assess portfolio active risk.

6 Conclusion

Among active equity managers, it is commonly known that *ex post* active risk often exceeds the target tracking error specified by a risk model. We attribute this deviation to an additional source of active risk—the strategy risk. Measured as the standard deviation of IC, strategy risk is unique to each investment strategy conveying a strategy-specific risk profile. Furthermore, through analytical derivations, we show that a *consistent* estimator of active risk must incorporate strategy risk in conjunction with the risk-model tracking error. Consequently, we provide a practical extension to the Fundamental Law of Active Management: *ex ante* IR equal to the ratio of average IC to the standard deviation of IC. Additionally, we also demonstrate that IR depends not only on the strength of IC, but also on the correlation between IC and the dispersion of the risk-adjusted returns over time.

Empirical evidence shows that risk models systematically underestimate *ex post* active risk. It is reasonable to expect this, because, by definition, the risk-model risk only accounts for tracking error caused by risk factors and specific risks specified by a risk model. However, all active strategies are exposed to alpha factors, which must have explanatory power for cross-sectional returns beyond the power provided by the risk model. This cross-sectional correlation between the alpha factor and the actual returns introduces additional risk not embedded in the risk model. Equation (9) provides a way to capture both the risk-model risk and the strategy risk associated with alpha factors.

This fact alone does not imply the deficiency of a risk model, because the job of a risk model is to capture the majority of cross-sectional dispersion in security returns embedded in commonly specified risk factors. While it is plausible that a given risk model might be improved with additional risk factors, it is unrealistic to expect a risk model to include all possible fundamental factors in all possible variations, as is often the case when active equity managers search for alpha factors. Combining the risk-model risk and the strategy risk represents a reasonable and realistic solution to the issue.

Our empirical survey of commonly used quantitative equity strategies confirms our analytical results. The difference in strategy risk is often statistically significant. We also illustrate how to use strategy risk to recalibrate the risk-model tracking error so that the *ex ante* active risk reaches a target level. While more sophisticated methods to forecast strategy risk await further research, such a simple modification has already proven far superior to just using the risk-model risk alone.

In addition to these benefits, our analysis also enables practitioners to estimate the *ex ante* excess return and active risk more accurately, without the daunting task of optimized back tests. This is especially true for market neutral equity hedge fund strategies with fewer portfolio constraints because our risk-constrained optimization closely resembles those strategies. For long-only active strategies, or other kinds of strategies with more constraints, our estimation could be combined with those of Grinold and Kahn (2000) and Clarke *et al.* (2002) to provide a more realistic IR estimate. Finally, for active equity managers, our analytical framework can be applied in a number of ways to provide a rigorous risk specification of equity investment strategies in terms of diversification benefit across strategies and most importantly better portfolio IR. For example, Sorensen *et al.* (2003) illustrate a way to combine multiple alpha sources more efficiently in an unconditional framework to achieve the highest portfolio IR. Alternatively, we can also apply the analysis in a conditional framework to take advantage of certain market conditions through tactical rotations of active investment strategies. These rotation tactics can be grounded on careful examination of how the strategy excess return and the strategy risk respond to different macro-environment, market segments (style or sector), and seasonal influences.

Appendix A: optimal active weights and excess return

This appendix provides mathematical details of the results in Section 1 regarding the optimal active weights and the excess return.

The active weights is the solution of the following optimization problem: Maximize

$$\mathbf{f}_t' \cdot \mathbf{w}_t - \frac{1}{2}\lambda_t \cdot (\mathbf{w}_t' \cdot \mathbf{V}_t \cdot \mathbf{w}_t) \qquad \text{(A.1)}$$

subject to

$$\begin{aligned} \mathbf{w}_t' \cdot \mathbf{i} &= 0 \\ \mathbf{w}_t' \cdot \mathbf{B}_t &= 0 \end{aligned} \qquad \text{(A.2)}$$

The subscript t denotes the period. For clarity, we omit it from our notation hereafter. In Eqs. (A.1) and (A.2), $\mathbf{f} = (f_1, f_2, \ldots, f_N)'$ is the vector of alpha factors or forecasts of excess returns over an index at time t; $\mathbf{w} = (w_1, w_2, \ldots, w_N)'$ the vector of active weights against the index; $\mathbf{B} = (\boldsymbol{\beta}_1, \boldsymbol{\beta}_2, \ldots, \boldsymbol{\beta}_M)$ the matrix of risk factors with each $\boldsymbol{\beta}_i$ a vector of risk factor; $\mathbf{i} = (1, 1, \ldots, 1)'$ the vector of ones; λ the risk-aversion parameter; and $\mathbf{V}$ the covariance matrix. The number of risk factors is M.

The covariance matrix $\mathbf{V}$ in a multi-factor risk model takes a special form:

$$\mathbf{V} = \mathbf{B} \cdot \boldsymbol{\Sigma}_\mathbf{B} \cdot \mathbf{B}' + \mathbf{S} \qquad \text{(A.3)}$$

where $\boldsymbol{\Sigma}_\mathbf{B}$ is the covariance matrix of risk factors, and $\mathbf{S} = \text{diag}(\sigma_1^2, \sigma_2^2, \ldots, \sigma_N^2)$ is the diagonal matrix of stock-specific risks. Equation (A.3) assumes zero correlation between stock-specific risks. Because we require that the active weights are factor neutral, and there is no systematic risk in the active weights whatsoever, we can reduce the objective function (A.1) to the following, provided that we keep all the constraints

$$\mathbf{f}' \cdot \mathbf{w} - \frac{1}{2}\lambda \cdot (\mathbf{w}' \cdot \mathbf{S} \cdot \mathbf{w}) \qquad \text{(A.4)}$$

We can now solve the optimization of (A.4) with the constraints (A.2) analytically using the method of Langrangian multipliers. We switch from matrix notation to using summations. The new objective function including $M + 1$ Langrangian multipliers (1 for the dollar neutral constraint and M for M risk factors) is

$$\begin{aligned} &\sum_{i=1}^{N} f_i w_i - \frac{1}{2}\lambda \sum_{i=1}^{N} w_i^2 \sigma_i^2 - l_1 \sum_{i=1}^{N} w_i \\ &\quad - l_2 \sum_{i=1}^{N} w_i \beta_{1i} - \cdots - l_{M+1} \sum_{i=1}^{N} w_i \beta_{Mi} \end{aligned} \qquad \text{(A.5)}$$

Taking the partial derivative with respect to w_i and equating it to zero gives

$$w_i = \lambda^{-1}\frac{f_i - l_1 - l_2\beta_{1i} - \cdots - l_{M+1}\beta_{Mi}}{\sigma_i^2} \quad \text{(A.6)}$$

The values of Langrangian multipliers are determined by the constraints through a system of linear equations.

Given the active weights, the portfolio excess return is the summed product of the active weights and the actual excess returns

$$\begin{aligned}\alpha &= \sum_{i=1}^{N} w_i r_i \\ &= \lambda^{-1}\sum_{i=1}^{N}\frac{f_i - l_1 - l_2\beta_{1i} - \cdots - l_{M+1}\beta_{Mi}}{\sigma_i^2} r_i\end{aligned} \quad \text{(A.7)}$$

To arrive at Eq. (4) with the risk-adjusted forecast and the risk-adjusted return, we replace the return r_i by $r_i - k_1 - k_2\beta_{1i} - \cdots - k_{M+1}\beta_{M1i}$, where $(k_2, \ldots, k_{M+1})$ are the returns to M risk factors. This does not change the equation due to the constraints placed on the active weights. We choose the value of k_1 to make the risk-adjusted return mean zero. Defining

$$\begin{aligned}F_i &= \frac{f_i - l_1 - l_2\beta_{1i} - \cdots - l_{M+1}\beta_{Mi}}{\sigma_i} \\ R_i &= \frac{r_i - k_1 - k_2\beta_{1i} - \cdots - k_{M+1}\beta_{M1i}}{\sigma_i}\end{aligned} \quad \text{(A.8)}$$

Eq. (A.7) becomes Eq. (4).

We next calculate the residual variance or equivalently the risk-model tracking error as the sum of active weights squared times the specific variance. The active portfolio has no market risk within the risk model because the active weights are neutral to all risk factors. We have

$$\sigma^2_{\text{model}} = \sum_{i=1}^{N} w_i^2\sigma_i^2 = \lambda^{-2}\sum_{i=1}^{N} F_i^2 \quad \text{(A.9)}$$

The residual variance is, therefore, the sum of the risk-adjusted forecasts squared. Therefore,

$$\begin{aligned}\sigma_{\text{model}} &= \lambda_t^{-1}\sqrt{\sum_{i=1}^{N} F_{i,t}^2} \\ &= \lambda_t^{-1}\sqrt{N-1}\sqrt{[\text{dis}(\mathbf{F}_t)]^2 + [\text{avg}(\mathbf{F}_t)]^2} \\ &\approx \lambda_t^{-1}\sqrt{N-1}\,\text{dis}(\mathbf{F}_t)\end{aligned} \quad \text{(A.10)}$$

We have assumed that $\text{avg}(\mathbf{F}_t) \approx 0$ and this approximation is quite accurate in practice.

Appendix B: The information ratio

This appendix presents the exact results regarding the expected excess return and active risk. To obtain the expected excess return and active risk based on Eq. (7) we must find the expected value and variance of a product of two random variables. We use x and y to denotes IC and the dispersion of the risk-adjusted returns.

Elementary statistical calculation tells us that

$$E(xy) = \bar{x}\bar{y} + \rho\sigma_x\sigma_y \quad \text{(B.1)}$$

The barred variables are averages and σ denotes the standard deviation, and ρ is the correlation. Identifying IC as the variable x and the dispersion of the risk-adjusted returns as the variable y, we obtain the expected excess return as in Eq. (10).

We can also obtain the variance of x times y as

$$\begin{aligned}\text{Var}(xy) = {} & \sigma_x^2\sigma_y^2 + \rho^2\sigma_x^2\sigma_y^2 \\ & + \bar{x}^2\sigma_y^2 + \bar{y}^2\sigma_x^2\end{aligned} \quad \text{(B.2)}$$

When $\sigma_y/\bar{y} \ll 1$ and $\sigma_y/\bar{y} \ll \sigma_x/\bar{x}$, i.e. the coefficient of variation for the dispersion of the risk-adjusted returns is much less than 1 and much less than the coefficient of variation for IC, the variance can be approximated by

$$\mathrm{Var}(xy) = \bar{y}^2\sigma_x^2 \qquad \text{(B.3)}$$

This approximation justifies using Eq. (9) for the active risk.

Notes

[1] This problem has also been recognized by other practitioners. For example, Freeman (2002) notes that "if a manager is optimizing the long-short portfolio, he or she better assume that the tracking error forecast (of a risk model) will be at least 50 percent too low."

[2] Grinold (1994) proposed this alpha formula mainly for translating cross-sectional z scores into alpha inputs for an optimizer. While such a prescription holds true for a linear time series forecast model, it is not theoretically valid with cross-sectional z scores. We demonstrate in the paper, that such a prescription is not necessary in deriving IR. Furthermore, while it is necessary to use a risk model for individual securities in the mean-variance optimization to form the optimal portfolio, it is not necessary and perhaps overreaching to assume returns of individual securities follow the prescription of the risk model. Instead of such a normative approach, we take a descriptive one, making no explicit assumptions about the expected return of each security.

[3] Later in the paper, we will use the time series standard deviation as well. To avoid confusion we shall use dispersion when describing cross-sectional standard deviation and standard deviation when describing time series standard deviation.

[4] It is difficult to maintain a constant level of risk-model tracking error for all time. One often targets it within a narrow range to accommodate portfolio drift and changing risk model estimates.

[5] The variance of N such independent variables is a scaled chi-square distribution if their mean is zero. It can be proven that when N is large, the dispersion is close to unity, using the approximation of a chi-square distribution (Keeping, 1995).

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Keywords: Portfolio management; portfolio optimization; active risk