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ON THE FINANCIAL INTERPRETATION OF RISK CONTRIBUTION: RISK BUDGETS DO ADD UP

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Due to a lack of clear financial interpretation, there are lingering questions in the financial industry regarding the concepts of risk contribution. This paper provides as well as analyzes risk contribution's financial interpretation that is based on expected contribution to potential losses of a portfolio. We show risk contribution, defined through either standard deviation or value at risk (VaR), is closely linked to the expected contribution to the losses. In addition, for VaR contribution, our use of Cornish–Fisher expansion method provides practitioners an efficient way to calculate risk contributions of portfolios with non-normal underlying returns. Empirical evidences are provided with asset allocation portfolios of stocks and bonds.

Introduction

The concepts of risk contribution[1] and percentage contribution to risk are widely used in both risk management and risk budgeting practices, in the areas of asset allocation as well as active portfolio management (Litterman, 1996; Lee and Lam, 2001; Wander *et al.*, 2002; Winkelmann, 2004). For instance, it is a cornerstone of portable alpha strategy (Arnott, 2002; Kung and Pohlman, 2004). Despite their ubiquitous presence, however, questions have remained regarding their validity, due to both the simple belief that risks are non-additive and

a lack of financial intuition behind mathematical definitions of these concepts. For instance, Sharpe (2002) correctly argues that a mere mathematical decomposition of risk does not necessarily qualify as risk contribution. The argument is sensible because, risk contribution is often defined, with little economic justification, through a calculation involving marginal contribution to risk—partial derivative of risk with respect to underlying security weights. Sharpe (2002) further suggests rejecting the concept of risk contribution altogether, based on the fact that, risk in terms of either standard deviation or value at risk (VaR), is non-additive. Chow and Kritzman (2001) express a similar critical view toward risk budgeting while emphasizing the usefulness of marginal contribution to VaR because of its clear financial interpretation.

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The primary reason for such lingering doubts might be due to the fact that, the financial industry as a whole, has yet come to a firm grasp of its financial interpretation beyond the initial mathematical definition. For example, Grinold and Kahn (2000) interpret it as "relative marginal contribution to risk." However, their interpretation is simply a recast of the definition for marginal contribution to risk. Earlier, Litterman (1996) also interpret risk contribution in terms of marginal analysis.

Does risk contribution have an independent, intuitive financial interpretation? Do risk budgets add up to 100%? These questions are important to all investment professionals who must consider risk as one of the dimensions in investment decision-making process. In this paper we answer these questions by providing an intuitive financial interpretation for the risk contribution. We shall present analytic results, empirical examples, and relate to most recent academic research by others on this important topic.

1   Loss contribution and loss budgets

We arrive at the interpretation by considering loss contribution and percentage contribution to loss. One of the common pressing questions facing portfolio managers is: In the event of a sizable loss to a portfolio, what are the likely contributions to the loss from the portfolio's underlying components? The answer turns out to be strongly consistent with the risk contribution as currently defined.

The answer is given by conditional expectations of underlying securities returns, given the total portfolio return equal to the loss. We first assume all security returns are normally distributed and use standard deviation as the risk measure. Later in the paper, we use more general risk measure VaR when the normality assumption is relaxed.

1.1   Conditional expectation with normal distribution

For simplicity, we work with an existing portfolio consisted of just two securities, with weights w_1 and w_2 respectively. We also assume their return volatilities σ_1, σ_2 and correlation ρ. Then the standard deviation of the portfolio return is

$$\sigma = \sqrt{w_1^2\sigma_1^2 + w_2^2\sigma_2^2 + 2\rho w_1 w_2 \sigma_1 \sigma_2} \quad (1)$$

The percentage contribution to risk[2] is given as the weight times the marginal contribution to risk times the marginal contribution to risk, divided by the standard deviation

$$p_1 = \left(w_1 \frac{\partial \sigma}{\partial w_1}\right) \Big/ \sigma = \frac{w_1^2\sigma_1^2 + \rho w_1 w_2 \sigma_1 \sigma_2}{\sigma^2}$$

$$p_2 = \left(w_2 \frac{\partial \sigma}{\partial w_2}\right) \Big/ \sigma = \frac{w_2^2\sigma_2^2 + \rho w_1 w_2 \sigma_1 \sigma_2}{\sigma^2}$$

$$(2)$$

Note the sum of percentage contribution to risk is unity, i.e., $p_1 + p_2 = 1$. Also note the percentage contribution is zero if the weight is zero, i.e., $p_i = 0$ when $w_i = 0$. As a matter of fact, Eq. (2) gives rise to an alternative interpretation of the percentage contribution p_i—it is the ratio of the covariance between component return of security i and the portfolio return, to the total variance of the portfolio. Therefore, it is the beta of the component return versus the portfolio, and naturally the sum of the betas is unity. This beta-interpretation is more appealing since it is not necessarily associated with the marginal contribution. But nevertheless, it still does not lend itself explicitly to an economic reason.

To connect the beta-interpretation to an economic interpretation, we consider the notion of loss contribution. Suppose the portfolio suffered a loss of size L. What are the expected percentage contributions to the loss from the two securities? In statistical

terms, it is $c_i = E(w_i r_i | w_1 r_1 + w_2 r_2 = L)/L, i = 1, 2$, i.e., the expected contribution from the two securities divided by the loss L. By the theory of conditional distribution[3] (see Appendix A for details), we have

$$c_1 = \frac{w_1 \mu_1}{L} + p_1 \left(1 - \frac{w_1 \mu_1}{L} - \frac{w_2 \mu_2}{L} \right)$$

$$= p_1 + \frac{p_2 w_1 \mu_1 - p_1 w_2 \mu_2}{L} \overset{\triangle}{=} p_1 + \frac{D_1}{L}$$

$$c_2 = \frac{w_2 \mu_2}{L} + p_2 \left(1 - \frac{w_1 \mu_1}{L} - \frac{w_2 \mu_2}{L} \right)$$

$$= p_2 + \frac{p_1 w_2 \mu_2 - p_2 w_1 \mu_1}{L} \overset{\triangle}{=} p_2 + \frac{D_2}{L}$$

$$(3)$$

We have assumed the unconditional expected returns of two securities are μ_1 and μ_2, respectively. In the equation, p_1 and p_2 are percentage contributions to risk, or beta, as defined in Eq. (2). It is easy to see the sum of percentage contribution to loss is also unity, i.e., $c_1 + c_2 = p_1 + p_2 = 1$, since $D_2 = -D_1$. In Eq. (3), we have defined the term D_1 and D_2. We shall discuss it at length later in the paper.

Equation (3) shows the expected percentage contributions to loss bear close relationship to the percentage contributions to risk. In fact, there are three instances where the two become identical.

1.2 Three special cases

First, if the unconditional expected returns μ_1 and μ_2 are both zero, then the loss contributions and risk contributions are the same, i.e., $c_1 = p_1, c_2 = p_2$, for any loss L. This case applies to short investment horizons where we can assume the expected returns are zero. In practice, much risk management analyses are indeed focused on short horizons such as one-day or one-week.

The second case is when one security has zero weight; therefore its contribution to risk is zero.

This is a trivial case in which the remaining security accounts for one hundred percent of risk as well as one hundred percent of loss. However, this loss contribution remains approximately true if the security weight is small and the loss L is relatively large compared to the sub-optimality of the portfolio.

The third and more interesting case arises when $D_1 = p_2 w_1 \mu_1 - p_1 w_2 \mu_2 = 0$, or equivalently

$$\frac{w_1 \mu_1}{p_1} = \frac{w_2 \mu_2}{p_2} \qquad (4)$$

Equation (4) is the first order condition of marginal utility for an optimal mean-variance portfolio. Therefore, it implies that for optimal portfolios, percentage contribution to risk is equivalent to expected percentage contribution to portfolio's total expected return. In other words, risk budgets become the budgets of expected return. Sharpe (2002) discusses this property at length and he suggests that "risk-budgeting and-monitoring systems, are best viewed in terms of a budget of implied expected excess return and deviation from the budget." However, this equivalency is only true for mean-variance optimal portfolios. For a real world portfolio, which might not be optimal in the mean-variance sense, our interpretation of percentage contribution to risk still allows managers to estimate the likely contribution to a given loss.

In fact, an additional benefit of Eq. (3) is it allows us to estimate the impact of portfolios' *sub-optimality* measured by $D_i's$ on the percentage contribution to loss. For instance, if the allocation to security 1 is more than the mean-variance optimal weight, then $D_1 = p_2 w_1 \mu_1 - p_1 w_2 \mu_2 < 0$. This is because when the weight w_1 increases from the optimal weight, the increase in its risk contribution dominates its increase in the expected return contribution. Therefore, for a given loss L that carries a negative sign, the percentage contribution to loss c_1 will be greater than the percentage contribution to risk p_1, because D_1/L is positive.

We further note that, when the loss L far exceeds the sub-optimality measures, the loss contributions are approximately the risk contribution, i.e., $c_1 \cong p_1$ and $c_2 \cong p_2$. This observation is very relevant during financial crises when portfolio losses could be significantly higher than the expected returns. Consequently, loss contribution would be well captured by risk contribution during the crisis periods. On the contrary, during quiet periods when portfolio losses are relatively small, loss contribution, or simply ex post return attribution, is unlikely to bear any relationship to risk contribution at all! Therefore, these small loss events should not be used as reasons to dismiss the usefulness of risk contribution.

In summary, percentage contribution to risk can be interpreted as percentage contribution to a given loss of the total portfolio. The two are identical when expected returns are zeros or when the portfolio is mean-variance optimal. In other cases, the interpretation is appropriate when the given loss is large compared to the sub-optimality measure $D'_i s$.

2   Ex post analyses of a balanced portfolio

In this section, we compare our theoretical predictions with actual results of a balanced portfolio investing 60% in S&P 500 Index and 40% in Ibbotson's Long-term Government Bond Index. The monthly returns span from January 1926 to June 2004. Table 1 gives statistics of monthly return indices, as well as returns of the balanced portfolio. The monthly average returns of stocks and bonds are 0.98% and 0.46% respectively, the monthly standard deviations are 5.61% and 2.27%, and their correlation is 0.14. The balanced portfolio has an average return of 0.78% and a standard deviation of 3.61%. By Eq. (2), the percentage contributions to risk are 90.3% from stocks and 9.7% from bonds. Therefore, a typical balanced portfolio might be balanced in terms of capital allocation but it is highly unbalanced in terms of risk allocation.

Table 1 also shows the higher moments of returns. While skewness is close to zeros for all three return series, the excess kurtosis is significantly positive, which raises question about the normality of the returns. As a first step, we nevertheless use normality assumption and standard deviation in our analysis. Later in the paper we shall take into account of the higher moments when analyzing VaR contribution.

2.1   Contribution to losses

Table 2 shows the predicted percentage contributions to loss together with the realized values for the balanced portfolio, for losses above 3% and in several bins.[4] In each bin, we only report contribution from stocks.[5] Since Eq. (3) only gives the expectation for a point loss, we use the midpoint of each bin to calculate the predicted percentage contribution.

Table 1 Monthly return statistics of indices and the balanced portfolio.

	S&P 500	US LT Gvt	60/40 Portfolio
Avg Return	0.98%	0.46%	0.78%
Stdev	5.61%	2.27%	3.61%
Skewness	0.39	0.66	0.40
Kurtosis	9.58	5.09	7.64
Corr w/S&P 500	1.00	0.14	0.97

Table 2 Comparison of average and standard deviation of percentage contribution to loss from stocks of a 60/40 portfolio.

Loss (%)	Predicted c_1 (%)	Realized c_1 (%)	N	Predicted Std (%)	Realized Std (%)
−4 to –3	93.5	89.8	45	28.0	26.1
−5 to –4	92.8	92.7	23	21.0	20.7
−6 to –5	92.3	88.1	11	16.8	16.1
−7 to –6	92.0	99.5	9	14.0	18.7
−8 to –7	91.8	90.1	8	12.0	18.6
−19 to –8	91.3	102.4	12	10.5	12.3

As the loss increases, the predicted c_1 decreases from 93.5% to 91.3%, approaching percentage contribution p_1, i.e., 90.3%. For this particular portfolio, the range of percentage contribution to loss from stocks is narrow because the portfolio's sub-optimality measure D_1 is small at −0.11% compared to the losses under consideration. For this reason, we could have used the percentage contribution to risk of 90.3% for all the losses considered. Table 2 shows that on average the predicted value agrees with the realized percentage contribution to loss. But it fails to capture the ascending trend of the realized values. This is especially true for the last bin where the range of the loss is −19% to −8%, in which the predicted value is roughly 91% while the realized value is above 102%. We have made the range of this bin much wider because there are only a few historical losses with these sizes—the column N denotes the number of data points in each bin and the number becomes smaller as the loss increases. We can postulate two possible reasons for the poor accuracy of the predicted c_1 at the tail end. One is we simply don't have enough data points, which might also explain the low value of the realized percentage contribution to loss in the loss bin from −8% to −7%. The other is the stock returns have fat tails. The latter is certainly consistent with the fact that the realized percentage contribution to loss from the stocks is higher than the predicted value based on normality assumption. In the next section,

where we take into account the high moments of the return distribution, the prediction would indeed improve.

The last two columns of Table 2 compare the predicted and realized standard deviations of the percentage contribution (see Appendix A). When the losses are capped at −6%, the two values are quite close. As the losses grow, again, the predicted values underestimate the realized ones, possibly due to stocks' excess kurtosis.

2.2 Variation of the sub-optimality measure for asset allocation portfolios

For the 60/40 balanced portfolio risk contribution and loss contribution show little difference because the sub-optimality measure D_1 is small compared to the return standard deviation. It is worth exploring whether this is true for other asset allocation portfolios. We thus deviate from our main theme to study other stock/bond portfolios. Figure 1 plots the ratio of sub-optimality measure D_1 to portfolio's return standard deviation σ, versus stock weight in asset allocation portfolios. There are three points at which $D_1 = 0$. As we explained above, when the stock weight is at either 0% or 100%, D_1 is zero. The third point corresponds to the mean-variance optimal portfolio in which the stock weight
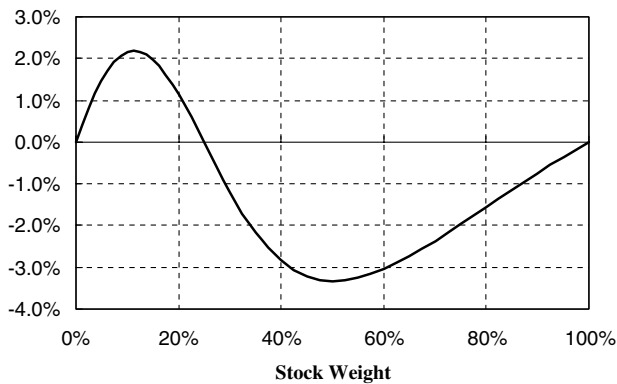
Figure 1 The value of D_1 over standard deviation for asset allocation portfolios.

is roughly 25%. When the stock weight is between 0% and 25%, D_1 is positive but as a percentage of return volatility is always less then 2%. When the stock weight is between 25% and 100%, D_1 is negative and the ratio to return volatility varies between 0% and -3%.

Therefore, it is true that D_1 is quite small compared to return standard deviation for all stock/bond portfolios. We conclude one can substitute risk contribution for loss contribution with little error, provided the loss is greater than one standard deviation. In other words, the financial interpretation of risk contribution or risk budget is accurately reflected for these asset allocation portfolios.

3   Percentage contribution to VaR

In this section, we extend the same financial interpretation to VaR contribution. VaR, representing loss with a given cumulative probability, is a general risk measure that can be used for portfolios with any kind of return distributions. For a portfolio with normal distribution, VaR is simply expected return plus a constant multiple of standard deviation. For a non-normal distribution, a $(1 - \alpha)\%$ VaR is defined through the following

equation

$$\text{Prob}(r \leq \text{VaR}) = \int_{-\infty}^{\text{VaR}} p(r)dr = \alpha \qquad (5)$$

where $p(r)$ is the probability density of the return distribution and α is the cumulative probability of loss, typically set at 5% or 1%.

3.1   Contribution to losses

We address two questions regarding VaR. First, can we define percentage contribution to risk in terms of VaR? If we can, the follow-up question is can we extend the same financial interpretation to the percentage contribution to VaR?

The answer to both questions is yes. First, because VaR is a linear homogeneous function of weights (Litterman, 1996; Hallerbach, 2003), it is mathematically true that

$$\text{VaR} = \sum_{i=1}^{N} w_i \frac{\partial \text{VaR}}{\partial w_i} \qquad (6)$$

Therefore one can define VaR contributions as weights time marginal contribution to VaR, which is sometimes referred to as component VaR. Dividing Eq. (6) by VaR leads to percentage contribution to VaR, or VaR-beta. Regarding the second question, Hallerbach (2002) gave an elegant proof that the VaR contribution given by Eq. (6) is exactly the expected contribution to a loss whose size equals to VaR (in Appendix B we give an alternative proof). Thus, VaR contribution can be interpreted as loss contribution.

While contributions to risk in terms of both standard deviation and VaR have the same financial interpretation, there are several subtle differences. First, in the case of standard deviation under normality assumption, percentage contributions to risk are independent of losses. We have shown under some circumstances, they approximate loss contri-

butions with sufficient accuracy regardless of the loss size. However, the interpretation of contribution to VaR is rather restrictive—it only applies to the loss that exactly equals specific VaR. VaR contributions change when VaR changes, i.e., they are different for 1% and 5% VaR. Therefore, for losses of different sizes, one must recalculate its VaR contribution. The second difference is their computational complexity. Whilst risk contribution based on standard deviation is easy to calculate, it is a daunting task to calculate risk contribution to VaR because analytic expressions are rarely available for VaR as a function of weights. Even when there is an analytic expression, calculating its partial derivative with respect to weights can be quite challenging (Chow and Kritzman, 2001; Chow *et al.*, 2001). In most instances, one has to resort to Monte Carlo simulations to obtain VaR decomposition as well as VaR itself. Fortunately, there exist analytic approximations to VaR based on moments of the return distribution, which in turn can also be used to estimate VaR contribution.

3.2   Analytic approximation to VaR decomposition

One such VaR approximation is based on Cornish–Fisher expansion (Mina and Ulmer, 1999). It retains the form of the mean plus a z score times the standard deviation

$$\text{VaR} = \mu + \tilde{z}_\alpha \sigma \qquad (7)$$

But the z-score $\tilde{z}_\alpha$ incorporates the normal z-score z_α and correction terms of higher moments

$$\tilde{z}_\alpha \approx z_\alpha + \frac{1}{6}\left(z_\alpha^2 - 1\right)s + \frac{1}{24}\left(z_\alpha^3 - 3z_\alpha\right)k$$
$$- \frac{1}{36}\left(2z_\alpha^3 - 5z_\alpha\right)s^2 \qquad (8)$$

Equation (8) gives an approximation of the α-percentile of a distribution with mean μ, standard deviation σ, skewness s, and excess kurtosis k. Substituting it into Eq. (7) yields the $(1 - \alpha)\%$ VaR.

As a numerical example, we calculate the 99% VaR of the 60/40 balanced portfolio using the two equations and inputs from Table 1. For the 99% VaR, we choose $\alpha = 0.01$ and then $z_\alpha = -2.33$, and from Eq. (8) we obtain $\tilde{z}_\alpha = -3.81$. Substituting it into Eq. (7) yields -12.96% for the 99% VaR. As a comparison, the 99% VaR without considering the higher moments is only -7.62%. Therefore the excess kurtosis of the return distribution dramatically increases the VaR at 99% level.

Previous researches (Mina and Ulmer, 1999; Jaschke, 2002) have shown that the Cornish–Fisher approximation provides an efficient and sufficiently accurate way to obtain VaR. To our knowledge, we are the first to use it as a way to approximate VaR decomposition analytically. The calculation is straightforward but laborious. VaR given by Eqs. (7) and (8) is an algebraic function of security weights—the mean is a linear function of weights, the standard deviation is the square root of a quadratic function of weights, the skewness and the kurtosis involve respectively third and fourth order polynomials of weights, with coefficients being third and fourth order co-moments between securities. For a given portfolio with multiple securities, once we have estimated the covariances and the higher order co-moments, for example, based on historical returns, Eq. (7) becomes an explicit but cumbersome function of the security weights. We can then derive analytically its partial derivatives with respect to weights and contributions to VaR. Details are omitted but are available upon request.

We present percentage contribution to VaR for our balanced portfolio in Table 3. As seen from Table 1, the skewness of the portfolio is only slightly positive,[6] but the excess kurtosis is significantly positive. Table 3 reports the percentage contributions to different values of VaR. These values are chosen to coincide with the midpoints of loss ranges

Table 3 Predicted percentage contribution to VaR compared to prediction based on normal distribution and realized percentage contribution to losses.

Loss (%)	Predicted VaR (%)	Predicted c_1 (%)	Realized c_1 (%)
−3.50	84.90	93.5	89.8
−4.50	90.50	92.8	92.7
−5.50	94.20	92.3	88.1
−6.50	97.10	92.0	99.5
−7.50	99.20	91.8	90.1
−8.50	100.90	91.3	102.4

in Table 2, so that we can compare the results directly with the predictions made under normality assumption and the realized values.

We make a few observations about the results. First, the predicted contribution from stocks to VaR increases as VaR increases, consistent with the general trend in the realized contribution to VaR. We have expected this because the stock returns have a higher kurtosis than the bond returns. In contrast, the prediction based on standard deviation decreases, albeit slightly, as the loss increases. For this reason, the VaR approach offers a better risk contribution analysis when the loss is large, above two standard deviations in this case. Second, when the loss is between one and two standard deviations, the standard deviation approach is competitive, possibly because in this range the effect of high kurtosis has yet been felt. Third, we note that while the VaR approach offers a better estimation for the expected contribution to loss, it does not provide a measure for the standard error, which, on the other hand, is readily available for risk contribution with standard deviation.

4   Conclusion

Both theoretical proof and empirical evidence show that risk contribution has a sound economic interpretation—expected contribution to potential losses of a portfolio. When the underlying return distributions are normal, risk contribution in terms of standard deviation is easy to calculate and often depict adequately the loss contribution. Furthermore, it provides error estimation for the loss contribution. Risk contribution in terms of VaR, on the other hand, is precise in theory, but hard to compute in practice. We show that the commonly-used Cornish–Fisher expansion method for VaR approximation can also be employed to estimate VaR contribution.

When both approaches are applied to the balanced portfolio, we obtain predictions that are in general agreement with the actual results. While the standard deviation approach is adequate when the loss is confined to two standard deviations, the VaR approach proves to be more accurate in the extreme tail since it accounts for the effect of higher moments.

In addition to risk management, our results have important implications to risk budgeting practice. First and foremost, risk budgets do add up. Second, risk budget for the same portfolio will look different depending which approach is used. In case of standard deviation approach, the budget could just be percentage contribution to risk since in many situations, the size of the loss does not alter the budget a great deal. But, in case of VaR approach, the budget picture depends on the percentile of VaR. A

95% VaR and 99% VaR could lead to very different risk budgets. Therefore, selecting a combination of portfolio VaR and its associated VaR contributions becomes an integrated task of risk budgeting exercise.

Finally, it is worth pointing out the Cornish–Fisher expansion method should very useful for obtaining VaR contribution for other types of asset allocation portfolios, especially those including hedge funds, whose returns could have significant skewness and kurtosis. Risk budget without considering these high moments could seriously underestimate the risk in the hedge funds. The Cornish–Fisher expansion method provides an efficient way to tackle this problem.

Acknowledgment

Appendix A.   Conditional expectation and conditional variance

In this appendix we present general results concerning percentage contribution to loss of a portfolio with N assets, whose joint return distributions are normal. The results include both conditional expectations and conditional variances. The case with two assets in the main text is a specific example.

We denote the joint return as a multivariate normal distribution, i.e., $\mathbf{r} \sim N(\mathbf{\mu}, \mathbf{\Sigma})$. In the distribution, $\mathbf{r} = (r_1, r_2, \ldots, r_N)'$ represents the random return vector, $\mathbf{\mu} = (\mu_1, \mu_2, \ldots, \mu_N)$ represents the means, and $\mathbf{\Sigma} = (\sigma_{ij})_{i,j=1}^{N} = (\rho_{ij}\sigma_i\sigma_j)_{i,j=1}^{N}$ is the covariance matrix. We specify the portfolio weights as $\mathbf{w} = (w_1, w_2, \ldots, w_N)'$, and the loss as L. Furthermore, we denote the portfolio return by $R = \mathbf{w}' \cdot \mathbf{r}$ and its mean by $\mu_R = \mathbf{w}' \cdot \mathbf{\mu}$. Then the expected percentage contribution to the loss is

$$
\begin{aligned}
c_i &= \frac{\mathrm{E}(w_i r_i | R = L)}{L} \\
&= \frac{w_i \mu_i}{L} + \frac{\mathrm{cov}(w_i r_i, R)}{\mathrm{var}(R)}\left(1 - \frac{\mu_R}{L}\right) \\
&= p_i + \frac{w_i \mu_i - p_i \mu_R}{L} \overset{\triangle}{=} p_i + \frac{D_i}{L} \quad (\text{A.1})
\end{aligned}
$$

In the equation, p_i is percentage contribution to risk from security i, the same as the beta of its component return versus the portfolio return. Several remarks can be made regarding the result. First, when the means are all zero, the expected contributions to loss are always identical to risk contribution p_i, regardless of the size of the loss. Second, this is also true for non-zero means if the portfolio is mean-variance optimal, i.e., if

$$
\frac{w_1 \mu_1}{p_1} = \frac{w_2 \mu_2}{p_2} = \cdots = \frac{w_N \mu_N}{p_N} \quad (\text{A.2})
$$

The portfolio's *sub-optimality* can be measured by the constants $D_i's$. And when they are small compared to the loss, the contribution to the loss is approximated by risk contribution p_i.

Using conditional distribution of normal variables, we also obtain the conditional standard deviation of percentage contribution to loss. We have

$$
s_i = \frac{\mathrm{std}(w_i r_i | R = L)}{L} = \frac{\sqrt{w_i^2 \sigma_i^2 - p_i^2 \sigma^2}}{L} \quad (\text{A.3})
$$

The numerator is the square root of the conditional variance of the return component from security i, which equals the unconditional variance minus beta squared times the variance of the total portfolio. As the loss L increases, the conditional standard deviation thus decreases as 1 over L.

For a portfolio with two securities, the two conditional standard deviations are the same

$$s_i = \frac{|w_1 w_2| \sigma_1 \sigma_2 \sqrt{1 - \rho^2}}{\sigma L}, \quad i = 1, 2 \quad \text{(A.4)}$$

Equation (A.4) is used to derive the results in Table 2.

Appendix B.   VaR decomposition

In this appendix, we show contribution to risk in terms of VaR has the same financial interpretation as expected contribution to loss. The VaR decomposition is defined through the following identity

$$\text{VaR} = \sum_{i=1}^{N} w_i \frac{\partial \text{VaR}}{\partial w_i} \quad \text{(B.1)}$$

Equation (B.1) holds true because VaR is a linear homogeneous function of weights. The VaR contribution or VaR decomposition is thus weight times the partial derivative of VaR with respect to the weight.

There are at least two ways to arrive at the financial interpretation. First, Hallerbach (2002) realizes that VaR, when considered as a portfolio return, can be written as a sum of conditional expectations of component returns

$$\text{VaR} = \sum_{i=1}^{N} w_i E(r_i | \mathbf{w}' \cdot \mathbf{r} = \text{VaR}) \quad \text{(B.2)}$$

Comparing Eqs (B.1) and (B.2) leads to

$$c_i = \frac{E(w_i r_i | \mathbf{w}' \cdot \mathbf{r} = \text{VaR})}{\text{VaR}} = \frac{w_i \frac{\partial \text{VaR}}{\partial w_i}}{\text{VaR}} \quad \text{(B.3)}$$

Equation (B.3) is the interpretation we have sought after—percentage contribution to VaR equals percentage contribution to a loss of the size given by VaR.

We next prove this result by a direct parametric approach. Without loss of generality, we again assume a portfolio of two securities, whose returns have a joint probability distribution $f(r_1, r_2)$ of rather arbitrary form. Then the cumulative probability of the portfolio return $R = w_1 r_1 + w_2 r_2$ being less than VaR is

$$\text{Prob}(R \leq \text{VaR})$$
$$= \int_{-\infty}^{\infty} dr_1 \int_{-\infty}^{(\text{VaR} - w_1 r_1)/w_2} f(r_1, r_2) dr_2 = \alpha.$$
$$\text{(B.4)}$$

Equation (B.4) defines the $(1 - \alpha)\%$ VaR as an implicit function of weights w_1, w_2. To obtain the partial derivative of VaR with respect to weights, we differentiate both sides of Eq. (B.4) with respect to w_1. We have

$$\int_{-\infty}^{\infty} f\left(r_1, \frac{\text{VaR} - w_1 r_1}{w_2}\right) \frac{1}{w_2}$$
$$\times \left(\frac{\partial \text{VaR}}{\partial w_1} - r_1\right) dr_1 = 0. \quad \text{(B.5)}$$

Therefore

$$w_1 \frac{\partial \text{VaR}}{\partial w_1} = \frac{\frac{1}{w_2} \int_{-\infty}^{\infty} w_1 r_1 f\left(r_1, \frac{VaR - w_1 r_1}{w_2}\right) dr_1}{\frac{1}{w_2} \int_{-\infty}^{\infty} f\left(r_1, \frac{VaR - w_1 r_1}{w_2}\right) dr_1}$$
$$\text{(B.6)}$$

The right hand side of this equation can be recognized exactly as the conditional expectation of $w_1 r_1$ given the portfolio return $R = w_1 r_1 + w_2 r_2 = \text{VaR}$. Thus, it established the financial interpretation for the VaR contribution.

Notes

[1] Risk contribution is often also called risk decomposition, or risk budget, among other names. Dividing the risk contribution by the total risk yields relative contribution or percentage contribution to risk. These are definitions we adhere to in this paper.

2 From this point on, we will only consider the percentage contribution, because the two measures are linearly dependent.

3 The conditional expectation of a normal variable equals to the unconditional mean plus its beta to the given variable, in this case, the total portfolio return, times the difference between the given variable and its unconditional mean.

4 We consider losses of the size close to and beyond the standard deviation of the portfolio.

5 Since the two percentage contributions add up to 100%, the bond contributions are omitted. Furthermore, the standard deviation of bonds' contribution equals to that of stocks.

6 For this reason, the last term in Eq. (8) is negligible thus is omitted from our calculation.

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